

Majorization properties of generalized thermal distributions

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Abstract

We examine the majorization properties of general thermal-like mixed states depending on a set of parameters. Sufficient conditions which ensure the increase in mixedness, and hence of *any* associated entropic form, when these parameters are varied, are identified. We then discuss those exhibiting a power law distribution, showing that they can be characterized by two distinct mixing parameters, one associated with temperature and the other with the non-extensivity index q . Illustrative numerical results are also provided.

Key words: Generalized thermal states, mixedness, majorization

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The rigorous concept of disorder derived from the theory of majorization [1,2,3,4] has recently received renewed attention in theoretical physics, particularly in the field of quantum information [5,6,7,8,9]. The essential reason is that it is stronger than that based on standard entropic considerations. The basic idea is that a given probability distribution or density matrix can be said to be “more mixed” or “disordered” than another only when it is *majorized* by the latter. This implies a higher entropy of the former, although the converse implication is not necessarily true. Majorization provides a natural partial ordering on probability distributions [2] and has consequently found many applications not only in mathematics, but also in other areas such as economy and computer science. Moreover, majorization relations are often naturally satisfied. For instance, in discrete classical systems, the joint distribution of two random variables is always majorized by the marginal distributions, while in quantum mechanics, the global density matrix ρ of a *separable* (i.e., non-entangled) mixed state of a composite system is always more mixed than the local reduced densities ρ_i of each subsystem [7], a property which can be violated by entangled states. These statements are stronger than the corresponding entropic inequalities [i.e., $S(\rho) > S(\rho_i)$].

The aim of this work is to examine along the previous lines the majorization properties of general thermal-like mixed states depending on a set of parameters, discussing as application those characterized by a power-law distribution [10,11,12]. The latter have in recent years been analyzed and successfully employed in a wide range of contexts [13,14,15,16,17], and can be derived within a generalized nonextensive thermodynamic formalism based on the Tsallis entropy [10]. One of the basic physical questions we want to answer is if such states *do become more disordered, in the way determined by majorization*, when the temperature (or some other fundamental parameter characterizing the distribution) is increased, as occurs with the standard Boltzmann-Gibbs thermal state. Such property would have far reaching consequences, in particular that of ensuring a *universal* entropy increase, i.e., an increase in *any* consistent disorder measure, and not just in that employed in the construction of the state. It would also imply the increase of the expectation value of *any* increasing function of energy, and not just of the energy itself. Here we will prove that such property is indeed valid.

For this purpose, we first identify the sufficient conditions that ensure the increase in mixedness of a general thermal-like mixed state when the parameters that characterize it are varied. We then show that states exhibiting a power-law distribution can be characterized by *two* distinct mixing parameters, one associated with temperature and the other with the non-additivity index q . We also discuss the majorization properties of escort distributions and the mixing conditions for generalized thermal-like states in the presence of constraints on non-commuting observables. Illustrative numerical results for a simple model are provided as well. Distributions with correct mixing properties can then be employed to investigate the effects of randomness in physical systems in a more complete way than that provided by the standard thermal state, allowing to explore different paths from the ground state to the full random state, all characterized by a proper disorder increase.

We will consider a quantum system of finite dimension n . The eigenvalues p_i of any density matrix ρ for such system ($p_i \geq 0$, $\sum_{i=1}^n p_i = 1$) will be sorted in what follows in *decreasing* order ($p_i \geq p_j$ if $i < j$). A density ρ is then said to be *more mixed* than a second density ρ' ($\rho \prec \rho'$) if the eigenvalues of ρ are *majorized* by those of ρ' [2,3,4]:

$$\rho \prec \rho' \Leftrightarrow s_j \equiv \sum_{i=1}^j p_i \leq s'_j \equiv \sum_{i=1}^j p'_i, \quad j = 1, \dots, n-1, \quad (1)$$

with $s_n = s'_n = 1$. In such a case, the probabilities p_i are more “spread out” than the p'_i ’s, and can be written as a convex combination of permutations of the latter, i.e., $\mathbf{p} = \sum_{\alpha} q_{\alpha} P_{\alpha}(\mathbf{p}')$, where P_{α} are permutations and $q_{\alpha} \geq 0$, $\sum_{\alpha} q_{\alpha} = 1$ [2,3,4]. The state described by ρ is then more mixed or “random”

than that described by ρ' . Accordingly, the completely random state $\rho = I/n$ (with I the identity) is more mixed than any density, while any density is more mixed than a pure state, i.e., $I/n \prec \rho \prec |\Phi\rangle\langle\Phi| \forall$ normalized density ρ and pure state $|\Phi\rangle$. It can be also shown that if $\rho \prec \rho'$, ρ can be written as a convex combination of unitary transformations of ρ' , i.e., $\rho = \sum_{\alpha} q_{\alpha} U_{\alpha}^{\dagger} \rho' U_{\alpha}$, with $q_{\alpha} > 0$ and $U_{\alpha}^{\dagger} U_{\alpha} = I$, and viceversa (Uhlmann's theorem [3]). If the dimensions of ρ and ρ' differ, the same definition (1) can be applied after completing with zeros the set of eigenvalues of the density of lowest dimension.

Let us briefly discuss now the relation with entropy. Consider for instance the general entropic forms [18,19,20,21]

$$S_f(\rho) = \text{Tr} f(\rho) = \sum_{i=1}^n f(p_i), \quad (2)$$

where f is a smooth strictly concave function ($f'(p_i) < f'(p_j)$ if $p_i > p_j$) defined in the interval $[0, 1]$, satisfying $f(0) = f(1) = 0$. The von Neumann entropy $S(\rho) = -\text{Tr} \rho \ln \rho$ and the Tsallis generalization [10],

$$S_q(\rho) = \text{Tr}(\rho - \rho^q)/(q - 1), \quad q > 0, \quad (3)$$

which approaches the von Neumann entropy for $q \rightarrow 1$, are the most important examples. It can be shown that if $\rho \prec \rho' \Rightarrow S_f(\rho) \geq S_f(\rho')$ for *any* f of the previous form [3,22] (the same holds for the Renyi entropy $S_q^R = \ln[1 + (1 - q)S_q(\rho)]/(1 - q)$ [23], since it is an *increasing* function of $S_q(\rho)$, as well as for any Schur concave function of ρ [4]). However, for a given f , the converse is not necessarily true, so that the concept of disorder implied by Eqs. (1) is stronger than that based on a particular choice of f . Nonetheless, the converse holds as follows: if $S_f(\rho) \geq S_f(\rho')$ for *any* f of the previous form $\Rightarrow \rho \prec \rho'$ [3,22]. In other words, the hallmark of increasing mixedness is a *universal* entropy increase. Note, however, that Eqs. (1) define a partial order relationship, in the sense that given two densities ρ, ρ' , it may happen that $\rho \not\prec \rho'$ and $\rho' \not\prec \rho$.

Let us now consider a general mixed state $\rho(\lambda)$ depending on a continuous parameter λ . We will say that λ is a *mixing parameter* in a certain interval if $\rho(\lambda)$ becomes *more mixed* as λ increases in this interval:

$$\rho(\lambda) \prec \rho(\lambda') \text{ if } \lambda \geq \lambda'. \quad (4)$$

This is equivalent, in the case of a smooth dependence, to the condition

$$\partial s_j / \partial \lambda \leq 0, \quad j = 1, \dots, n-1,$$

within this interval. The generalized entropy $S_f[\rho(\lambda)]$ is then a *non-decreasing*

function of λ for *any* concave f , as easily verified:

$$\frac{\partial S_f[\rho(\lambda)]}{\partial \lambda} = \sum_{j=1}^{n-1} \frac{\partial s_j}{\partial \lambda} [f'(p_j) - f'(p_{j+1})] \geq 0, \quad (5)$$

since $f'(p_j) - f'(p_{j+1}) \leq 0$ for f concave. Such states exhibit then an unambiguous disorder increase for increasing λ .

As a general example, let us consider the escort densities [23]

$$\rho_q = \rho^q / Z_q, \quad Z_q = \text{Tr } \rho^q, \quad (6)$$

associated with a density matrix ρ . It is easily seen that for $q > 0$, $\lambda = 1/q$ is a *mixing parameter* for ρ_q , since the ensuing partial sums satisfy, for $j < n$,

$$\frac{\partial s_j}{\partial \lambda} = -q^2 \sum_{i=1}^j \sum_{k=j+1}^n p_{qi} p_{qk} \ln(p_i/p_j) \leq 0, \quad \lambda = 1/q > 0, \quad (7)$$

where $p_{qi} = p_i^q / Z_q$ are the eigenvalues of ρ_q ($p_{qi} \geq p_{qj} \forall q > 0$ if $i \leq j$). Any other decreasing function of q is of course a mixing parameter for ρ_q as well. We have therefore $\rho_q \prec \rho_{q'}$ if $0 < q \leq q'$ for a given fixed density matrix ρ .

Assume now that the system is described by a Hamiltonian H with energies ε_i , $i = 1, \dots, n$, sorted in what follows in *increasing* order, and consider densities $\rho(\lambda)$ which satisfy the conditions: a) they *commute* with H , b) their eigenvalues are *non-increasing* functions of energy ($p_i \geq p_j$ if $\varepsilon_i \leq \varepsilon_j$) and c) λ is a mixing parameter for $\rho(\lambda)$ in a certain interval. In such a case, another fundamental consequence of Eq. (4) is that not only the average energy $\langle H \rangle_\rho = \text{Tr} \rho(\lambda) H$, but also the expectation value of *any* non-decreasing function w of H ($w(\varepsilon_i) \leq w(\varepsilon_j)$ if $i < j$), independent of λ , is a *non-decreasing* function of λ :

$$\frac{\partial \langle w(H) \rangle_\rho}{\partial \lambda} = \sum_{j=1}^{n-1} \frac{\partial s_j}{\partial \lambda} [w(\varepsilon_j) - w(\varepsilon_{j+1})] \geq 0. \quad (8)$$

This automatically ensures a non-negative generalized “specific heat” $c_\lambda \equiv \partial \langle H \rangle_\rho / \partial \lambda \geq 0$. Let us also remark that if $\partial s_j / \partial \lambda$ were positive for some j (and $p_j > p_{j+1}$, $\varepsilon_j < \varepsilon_{j+1}$), one could always find functions f and w of the previous forms such that Eqs. (5) and (8) become negative. In this way, one can in principle always witness the absence of proper mixing increase.

We may now say that $\rho(\lambda)$ exhibits a *thermal-like behavior* if in addition, it approaches the ground state density I_1/n_1 in some limit $\lambda \rightarrow \lambda_0$ (I_1 denotes the projector onto the ground state energy subspace and n_1 its degeneracy)

and the state of maximum disorder I/n in some other limit $\lambda \rightarrow \lambda_\infty$, with λ a mixing parameter for $\lambda_0 < \lambda < \lambda_\infty$. The most common example of a state of the previous form is, of course, the standard Boltzmann-Gibbs (BG) thermal state (we set in what follows Boltzmann constant $k = 1$)

$$\rho(T) = \exp[-H/T]/Z(T), \quad Z(T) = \text{Tr} \exp[-H/T], \quad T > 0, \quad (9)$$

obtained from the minimization of $\langle H \rangle_\rho - TS(\rho)$. It is well known that its von Neumann entropy $S[\rho(T)]$ is an increasing function of temperature, which is usually taken as the basis for the statement that $\rho(T)$ becomes more disordered as T increases. However, in the present framework this statement can be more rigorously formulated. It is easily shown that T is a *proper mixing parameter* for $\rho(T)$ in the interval $(0, \infty)$, i.e.,

$$\rho(T) \prec \rho(T') \quad \text{if } T \geq T' > 0, \quad (10)$$

as the sums of its first j eigenvalues satisfy, for $j = 1, \dots, n-1$,

$$\frac{\partial s_j}{\partial T} = \sum_{i=1}^j \sum_{k=j+1}^n p_i p_k (\varepsilon_i - \varepsilon_k) / T^2 \leq 0. \quad (11)$$

Hence, not only its von Neumann entropy, but also its *generalized entropy* $S_f[\rho(T)]$, is an increasing function of T for any concave f . The average of any increasing function of energy is an increasing function of T as well. It can be shown that the generalized thermal density obtained from the minimization of $\langle H \rangle_\rho - TS_f(\rho)$, given by [20] $\rho_f(T) = f'^{-1}(H/T + \alpha)$, where α is a normalization constant and the cutoff $f'^{-1}(u) = 0$ if $u > f'(0)$ applies, also becomes more mixed as T increases [17]. Eq. (11) remains valid replacing $p_i p_k$ by $\tilde{Z} \tilde{p}_i \tilde{p}_k$, with $\tilde{p}_i = -[f''(p_i) \tilde{Z}]^{-1}$ for $p_i > 0$ and $\tilde{Z} = -\sum_{p_i > 0} [f''(p_i)]^{-1}$, being both positive for f concave.

Let us now identify the conditions which ensure that λ is a mixing parameter for a density of the more general form

$$\rho(\lambda) = g(H, \lambda) / Z(\lambda), \quad Z(\lambda) = \text{Tr } g(H, \lambda), \quad (12)$$

where $g(\varepsilon, \lambda)$ is assumed to be an arbitrary smooth positive non-increasing function of ε for $\varepsilon \in [\varepsilon_1, \varepsilon_n]$ depending on a parameter λ , with $[\rho(\lambda), H] = 0$. The variation rate of the associated partial sums can be shown to be, for $j < n$,

$$\frac{\partial s_j}{\partial \lambda} = \sum_{i=1}^j \sum_{k=j+1}^n p_i p_k [\tilde{g}_\lambda(\varepsilon_i) - \tilde{g}_\lambda(\varepsilon_k)], \quad \tilde{g}_\lambda(\varepsilon) \equiv \frac{\partial \ln g}{\partial \lambda}, \quad (13)$$

with $p_i = g(\varepsilon_i, \lambda)/Z(\lambda)$, which generalizes Eq. (11). A *sufficient* condition which ensures $\partial s_j / \partial \lambda \leq 0$ for $j = 1, \dots, n-1$ is, therefore, that $\tilde{g}_\lambda(\varepsilon)$ be a *non-decreasing* function of ε for $\varepsilon \in [\varepsilon_1, \varepsilon_n]$, i.e.,

$$\frac{\partial^2 \ln g}{\partial \varepsilon \partial \lambda} \geq 0 \Rightarrow \frac{\partial s_j}{\partial \lambda} \leq 0, \quad j = 1, \dots, n-1. \quad (14)$$

For instance, if

$$\rho(\lambda) = g(H/\lambda)/Z(\lambda), \quad (15)$$

where $\lambda > 0$ and g is here a positive non-increasing smooth real function, $\tilde{g}_\lambda(\varepsilon) = -\lambda^{-1}u[\ln g(u)]'$, with $u = \varepsilon/\lambda$, and Eq. (14) leads to the condition

$$-\{u[\ln g(u)]'\}' \geq 0, \quad (16)$$

which is not necessarily valid (consider for instance $g(u) = [1 + \ln(u+1)]^{-1}$ for $u > 0$). It is, of course, valid in the BG case for $\lambda = T$ ($g(u) = \exp[-u]$) as well as for $g(u) = \exp(-u^r) \forall r > 0$ if $u > 0$ ($-\{u[\ln g(u)]'\}' = r^2 u^{r-1} \geq 0$). Note also that if $g(0) > 0$ and $g(\infty) = 0$, the state $\rho(\lambda) = g(\bar{H}/\lambda)/Z(\lambda)$, with $\bar{H} = H - \varepsilon_1 I$, will always approach the ground state density I_1/n_1 for $\lambda \rightarrow 0$ and the fully mixed state I/n for $\lambda \rightarrow \infty$. It is, however, the more stringent condition (16) which ensures that $\rho(\lambda)$ will in addition become monotonously *more mixed* as it evolves from the ground state to the fully mixed state.

Note also that if λ is a mixing parameter for the density (12), it will remain a mixing parameter for the associated escort density (6) for $q > 0$, which corresponds to $g_q(H, \lambda) = g(H, \lambda)^q$. The sign of $\frac{\partial^2 \ln g_q}{\partial \varepsilon \partial \lambda}$ is left unchanged for λ independent of q . Eq. (7) also follows from Eq. (13) for $g(H, \lambda) \rightarrow g(H)^{1/\lambda}$.

As an important example of Eq. (12), we will examine the mixing properties of density operators characterized by a power-law distribution, which can be written in the form of the Tsallis distribution [10,11]

$$\rho(q, T^*) = [I - (1-q)\bar{H}/T^*]_+^{\frac{1}{1-q}} / Z(q, T^*), \quad (17)$$

where $T^* > 0$ represents an effective temperature, q the non-extensivity index, $\bar{H} = H - \varepsilon_1 I$ the energy measured from the ground state and $[u]_+ \equiv (u + |u|)/2$. For $q \rightarrow 1$, $\rho(q, T)$ approaches the BG distribution (9). Eq. (17) is obviously positive and fulfills previous conditions a) and b) $\forall T^* > 0$ and $q \in \mathfrak{R}$. Its eigenvalues p_i are strictly decreasing functions of energy for $q > 1$, but just non-increasing functions for $q < 1$ due to the *cutoff* that applies in this case ($p_i = 0$ if $(1-q)\bar{\varepsilon}_i \geq T^*$, where $\bar{\varepsilon}_i \equiv \varepsilon_i - \varepsilon_1$).

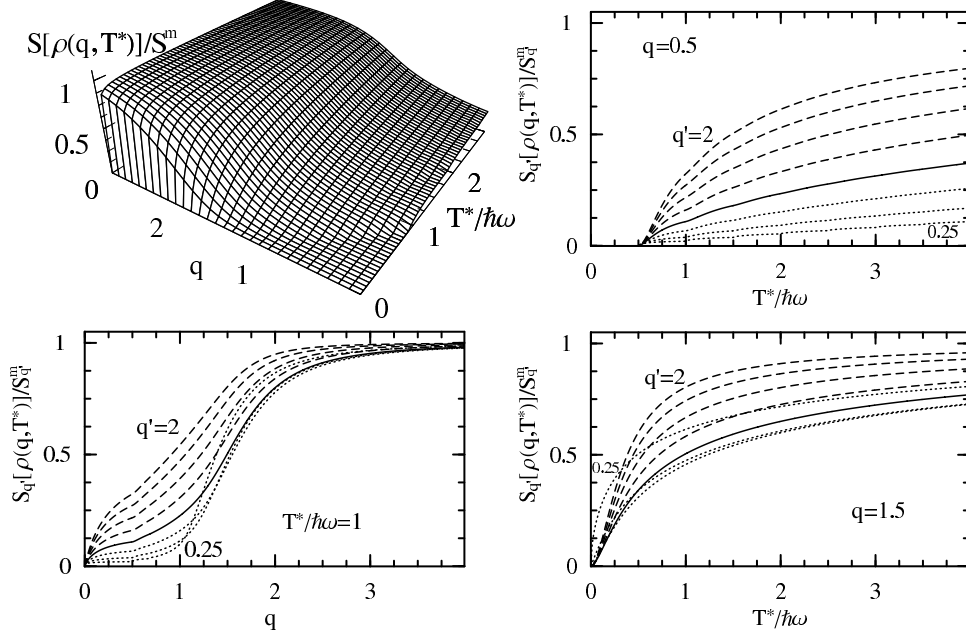


Fig. 1. Top left: The von Neumann entropy of the Tsallis distribution (17) (scaled to the maximum value $S^m \equiv S(I/n)$) for $n = 100$ equally spaced levels with spacing $\hbar\omega$, showing that it is a non-decreasing function of both T^* and q . Remaining panels: The scaled Tsallis entropy $S_{q'}[\rho(q, T^*)]/S_{q'}^m$ in the same system, as a function of T^* at fixed q (right panels) and as a function of q at fixed T^* (bottom left panel), for different values of q' , ranging from $q' = 0.25$ to $q' = 2$ in steps of 0.25. Dashed (dotted) lines correspond to $q' > 1$ ($q' < 1$), solid lines to $q' = 1$ (von Neumann entropy). They are all increasing functions of T^* and q .

We will now show that *both* T^* and q are *proper mixing parameters* for $\rho(q, T^*)$. Defining $g(\bar{\varepsilon}, q, T^*) \equiv [1 - (1 - q)\bar{\varepsilon}/T^*]_+^{\frac{1}{1-q}}$, we obtain

$$\frac{\partial^2 \ln g}{\partial \bar{\varepsilon} \partial T^*} = \frac{1}{(T^* - (1 - q)\bar{\varepsilon})^2} \geq 0, \quad \frac{\partial^2 \ln g}{\partial \bar{\varepsilon} \partial q} = \frac{\bar{\varepsilon}}{(T^* - (1 - q)\bar{\varepsilon})^2} \geq 0, \quad (18)$$

for $\bar{\varepsilon} > 0$ and $(1 - q)\bar{\varepsilon}/T^* < 1$, so that according to Eq. (14), $\rho(q, T^*)$ becomes *more mixed* as *either* T^* *or* q *increases*:

$$\rho(q, T^*) \prec \rho(q, T'^*) \text{ if } T^* \geq T'^* > 0, \quad \rho(q, T^*) \prec \rho(q', T^*) \text{ if } q \geq q'. \quad (19)$$

The role of T^* and q as proper mixing parameters constitutes then another fundamental property of the distribution (17). *Any* entropy $S_f[\rho(q, T^*)]$ (in particular $S_{q'}[\rho(q, T^*)]$ for *any* $q' > 0$) is a non-decreasing function of both T^* and q in *any* system, as illustrated in fig. 1 for a truncated harmonic oscillator.

Let us remark that Eq. (17) exhibits a *proper thermal-like behavior with respect to both* q *and* T^* , since in addition it approaches the ground state density I_1/n_1

both for $T^* \rightarrow 0$ at fixed q and for $q \rightarrow -\infty$ at fixed T^* , and the random state I/n both for $T^* \rightarrow \infty$ at fixed q as well as for $q \rightarrow \infty$ at fixed T^* . Actually, due to the cutoff for $q < 1$, $\rho(q, T^*) = I_1/n_1$ already for $T^* < (1 - q)\Delta$ at fixed $q < 1$, and $q < 1 - T^*/\Delta$ at fixed T^* , where Δ is the lowest non-zero excitation energy. Note also that $\rho(q, T^*) - I/n$ behaves *linearly* with \bar{H} for large T^* ($g(\bar{\varepsilon}, q, T^*) \approx 1 - \bar{\varepsilon}/T^*$ for $T^* \gg |1 - q|\bar{\varepsilon}$) but *logarithmically* for large q ($g(\bar{\varepsilon}, q, T^*) \approx 1 - \ln[1 + q\bar{\varepsilon}/T^*]/q$ for $q \gg \ln[1 + q\bar{\varepsilon}/T^*]$ and $q \gg 1$).

Eq. (17) can also be characterized by other mixing parameters. For example, rewriting Eq. (17) as $\rho_s(\gamma, \mu) = [1 - s\bar{H}/\mu]_+^{s/\gamma}/Z(\gamma, \mu)$, with $\gamma = |1 - q| > 0$, $\mu = T^*/|1 - q| > 0$ and $s = \text{Sign}(1 - q)$, *both* μ and γ are as well independent mixing parameters for both signs of s (despite the increase of γ with decreasing q for $q < 1$), since $\frac{\partial^2 \ln g}{\partial \bar{\varepsilon} \partial \mu} = [\gamma(\mu - s\bar{\varepsilon})^2]^{-1} \geq 0$, $\frac{\partial^2 \ln g}{\partial \bar{\varepsilon} \partial \gamma} = [\gamma^2(\mu - s\bar{\varepsilon})]^{-1} \geq 0$ for $s\bar{\varepsilon}/\mu < 1$. Thus, $\rho_s(\gamma, \mu) \prec \rho_s(\gamma, \mu')$ if $\mu \geq \mu'$ and $\rho_s(\gamma, \mu) \prec \rho_s(\gamma', \mu)$ if $\gamma \geq \gamma'$. These properties also follow from Eq. (16) in the case of μ (for $g(u) = [1 - su]_+^{s/\gamma}$) and from Eqs. (6)–(7) in the case of γ . Again, we obtain a thermal-like behavior with respect to both γ and μ for $s = \pm 1$, with the ground state density approached for $\mu \rightarrow 0$ or $\gamma \rightarrow 0$ and the full random state approached for $\mu \rightarrow \infty$ or $\gamma \rightarrow \infty$ (provided $\mu > \bar{\varepsilon}_n$ if $s = 1$).

The actual thermal state derived from the non-extensive thermodynamic formalism based on the Tsallis entropy and the minimization of the free energy $F_q = \langle H \rangle_{\rho_q} - TS_q(\rho)$ [12,15], where $q > 0$ and ρ_q is the escort density (6), is also of the form (17) but with T^* related to the actual T by $T = [T^* - (1 - q)\langle \bar{H} \rangle_{\rho_q}]/Z_q$ (following ref. [15]). For $q > 1$, T is a direct increasing function of T^* , as in this case $\partial T/\partial T^* = (\eta q - 1)/[(q - 1)Z_q] \geq 0$ (with $\eta = 2\langle O \rangle_{\rho_q} \langle O^{-1} \rangle_{\rho_q} - 1 \geq 1$ and $O = I - (1 - q)\bar{H}/T^*$), so that T will also be a *proper mixing parameter*. It will remain so for $0 < q < 1$ provided the absolute minimum of the free energy F_q at each T is considered [24], as in this case the entropy (and hence T^*) cannot decrease with increasing T .

So far all previous expressions are applicable in both the quantum and the classical discrete case. Let us finally briefly examine the majorization properties of density matrices constructed from constraints on two or more *non-commuting observables*. We may for instance consider two observables H_0, H_1 , with $[H_0, H_1] \neq 0$, and a density of the form

$$\rho(\lambda_0, \lambda_1) = g(H_0/\lambda_0 + H_1/\lambda_1)/Z(\lambda_0, \lambda_1), \quad (20)$$

where $g(u)$ is a positive non-increasing function and $\lambda_\nu > 0$ for $\nu = 0, 1$, generalizing Eq. (15). In this case, Eq. (13) should be replaced, for $j < n$, by

$$\frac{\partial s_j}{\partial \lambda_\nu} = \sum_{i=1}^j \sum_{k=j+1}^n p_i p_k [\tilde{g}_i^\nu - \tilde{g}_k^\nu], \quad \tilde{g}_i^\nu = \frac{\partial \ln g(u_i)}{\partial \lambda_\nu} = -\frac{g'(u_i)}{\lambda_\nu^2 g(u_i)} \langle i | H_\nu | i \rangle, \quad (21)$$

where u_i denotes the eigenvalues (sorted in increasing order) of $\sum_{\nu=0,1} H_{\nu}/\lambda_{\nu}$ and $|i\rangle$ the corresponding eigenstates (in case of degeneracy we assume in (21) H_{ν} diagonal within each eigenspace). Hence, we can ensure that λ_{ν} will be a mixing parameter for (20) if \tilde{g}_i^{ν} *does not decrease for increasing values of u_i* .

For a single observable H_0 , $\tilde{g}_i^0 = -\lambda_0^{-1} u_i g'(u_i)/g(u_i)$ and the previous condition reduces to Eq. (16). This is also the case when $\langle i|H_{\nu}|i\rangle$ is *proportional* to u_i ($\langle i|H_{\nu}|i\rangle = \alpha u_i$, with $\alpha > 0$ and independent of i), as occurs in simple systems such as a harmonic oscillator $H = [p^2/m + kx^2]/2$. In this case $\langle i|p^2|i\rangle = mE_i$, $\langle i|x^2|i\rangle = E_i/k$, with $E_i = \hbar\omega(i + 1/2)$ the oscillator energies ($\omega = \sqrt{k/m}$), so that both the *mass* m and the *inverse oscillator strength* k^{-1} are also *mixing parameters* in $\rho = g(H/T)/Z(T)$, provided T is a mixing parameter too (for $\lambda_0 = m$, $\lambda_1 = k^{-1}$, we would have $\tilde{g}_i^{\nu} = -\frac{1}{2}\lambda_{\nu}^{-1}[u_i g'(u_i)/g(u_i)]$, with $u_i = E_i/T$). In the general case, however, $\langle i|H_{\nu}|i\rangle/u_i$ may depend on i in a non-trivial way, so that the mixing properties of ρ will require a careful analysis of the behavior of \tilde{g}_i^{ν} .

In conclusion, we have applied the theory of majorization to identify the rigorous sufficient mixing conditions, as well as their main physical implications, for general mixed states of the form (12), which are summarized in Eqs. (14), (16), (5) and (8). As application, we have examined those characterized by a power law distribution, and shown that they can be expressed in terms of two fundamental mixing parameters, which can be taken as T^* and q in the representation (17). We have in particular identified the role of q in (17) as a rigorous mixing parameter. The actual thermal state derived in the Tsallis non-extensive thermodynamic formalism was also shown to become more mixed for increasing T (with the above remarks applying for $0 < q < 1$), as occurs with the standard BG thermal state, ensuring in particular a universal entropy increase, i.e., $\partial S_f[\rho(q, T)]/\partial T \geq 0$ for *any* concave f . These results strengthen thus the robustness of the generalized non-extensive thermodynamic formalism. We have also discussed the majorization properties of escort distributions [Eq. (6)] and derived sufficient conditions for mixedness increase in the presence of non-commuting observables [Eq. (21)].

Majorization theory enables then to derive very general inequalities with deep implications by simple means. Generalized thermal-like distributions with proper mixing parameters may also help to provide a more complete description of the behavior of a correlated quantum system with increasing randomness, revealing aspects which could be hidden in standard BG statistics [17]. For instance, any system possessing a limit temperature T_c in standard statistics for some property present in its ground state and absent in the vicinity of the completely random state, will also possess a limit value of the mixing parameter in a generalized thermal-like distribution. Knowledge of such boundaries (like a critical curve $T_c^*(q)$ in (17)) may provide a new perspective for the classification of order-disorder transitions or crossovers.

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